SEMINAR: SIMPLE HOMOTOPY THEORY AND WHITEHEAD GROUPS UNIVERSITÄT REGENSBURG (SS19, TUESDAY 10-12, M009)

GEORGIOS RAPTIS

1. Seminar Description

Algebraic topology is generally concerned with the notion of homotopy equivalence. This Seminar will be about a combinatorial approach to the definition of homotopy equivalence for finite CW complexes. We will ask the following fundamental question: is there is a collection of elementary geometrically defined homotopy equivalences from which every homotopy equivalence is generated?

The definition of simple homotopy equivalence is based on such elementary homotopy equivalences, which act as building blocks or series of moves. The *question* then becomes: is every homotopy equivalence between finite CW complexes *simple*?

The answer turns out to be "*no*", but in a very interesting way. There is a single algebraic invariant, called the *Whitehead torsion*, which decides whether a homotopy equivalence is simple. The Whitehead torsion of a homotopy equivalence is an element of an abelian group, called the *Whitehead group*, which depends only on the fundamental group. This algebraic invariant leads to a complete answer of the question and it establishes a connection between topology and algebra that is a beautiful (and rare) instance where these match up exactly.

Outline: We will first introduce and study simple homotopy equivalences of finite CW complexes. Then we will define Whitehead groups and discuss their algebraic properties. Then we will prove the relationship between the topological theory of simple homotopy equivalences and the algebraic properties of Whitehead groups, and answer the question stated above. Finally, we will discuss *lens spaces* which provide examples of homotopy equivalent finite CW complexes which are not simple-homotopy equivalent.

The Seminar should be of interest to those interested in algebraic topology. The topic can also serve as an introduction to algebraic K-theory from a topological viewpoint.

Prerequisites: Fundamental groups; Covering spaces; CW complexes; Singular and cellular homology.

Preliminary Reading: [1, §1 - §3].

2. Seminar Schedule

Simple Homotopy Equivalences

Talk 1 (30.04.2019): *Elementrary expansions and collapses. General properties.* Introduce the definitions of elementary expansion, elementary collapse, and simple homotopy equivalence. Give some examples and explain some basic properties of the simple-homotopy equivalence relation. Reference: [1, §4-§5].

Talk 2 (07.05.2019): *The (topological) Whitehead group.* Define the Whitehead group of a CW complex and prove that it is indeed a well–defined abelian group. Discuss the functoriality properties of this construction. Further properties of the simple-homotopy equivalence relation. Reference: [1, §5-§6-§7].

Talk 3 (14.05.2019): *Simplifying homotopically trivial CW pairs.* Explain the inductive procedure for simplifying a homotopically trivial CW pair to a 'normal/simplified form': a pair which has relative cells only in two consecutive dimensions. Identify the matrix associated to a homotopically trivial CW pair in simplified form. Reference: [1, §7-§8].

Talk 4 (21.05.2019): *Matrices and deformations.* Prove that the certain deformations of the matrix of a homotopically trivial CW pair in simplified form correspond to topological operations which do not change the simple-homotopy type. Reference: [1, §8].

Whitehead Torsion (Algebra)

Talk 5 (28.05.2019): *Whitehead groups.* Define $K_1(R)$ (algebraic K_1 of a ring R) and $Wh(G)$ (Whitehead group of a group G) and explain some of their basic properties. Give some examples of calculations. Reference: [1, §10-§11].

Talk 6 (04.06.2019): *Acyclic chain complexes.* Introduce the notion of a simple isomorphism. Define (R, G)-chain complexes, acyclic chain complexes, and stable equivalences of acyclic chain complexes. Prove the existence of a chain contraction and the independence of its choice for the purpose of defining the torsion. Reference: [1, §12-§13-§14].

Talk 7 (18.06.2019): *Whitehead torsion.* Define the torsion of an acyclic (R, G)-chain complex and give a characterization in terms of its properties. Reference: [1, §14-§15-§17].

Whitehead Torsion (Topology)

Talk 8 (25.06.2019): *The Whitehead torsion of a CW pair.* Define the Whitehead torsion of a homotopically trivial CW pair in detail and deduce some its basic properties. Reference: [1, §19-§20].

Talk 9 (02.07.2019): *Comparison between Whitehead groups.* Prove the equivalence between the topological and the algebraic definitions of the Whitehead group. Reference: [1, §21 - §22, §24].

Talk 10 (09.07.2019): *The product and sum theorems.* State and prove the product and sum theorems for the Whitehead torsion. State the topological invariance of the Whitehead torsion (without proof). Reference: [1, §23, §25, Appendix].

Example : Lens Spaces

Talk 11 (16.07.2019): *Lens spaces and homotopy classification.* Define lens spaces and their CW structures. Discuss the homotopy classification (in dimension 3). Reference: [1, §26-§28].

Talk 12 (23.07.2019): *Simple-homotopy classification of lens spaces.* Prove the simple-homotopy classification of 3-dimensional lens spaces. Reference: [1, §30].

References

[1] M. M. Cohen, *A Course in Simple-Homotopy Theory*, Graduate Texts in Mathematics No. 10, Springer-Verlag, 1973.

G. Raptis
Email address: georgios.raptis@mathematik.uni-regensburg.de